This connection has been exploited in other contexts (for other groups).

(2) $a_n(y,s)$ for $n \geq 1$ has the form.

$$a_n(y,s) = \underbrace{a(n,s)}_{\text{purely arithmetic part } \doteq |n|^{s-\frac{1}{2}} \sigma_{1-2s}|n|} \underbrace{W_s(|n|, y)}_{\text{special function of } y}$$

instead of the holomorphic case where the special function is simply $e^{-2\pi n y}$.

Pf. The proof is technical but straightforward.
Let $z \in \mathbb{H}$

$$G(z,s) = \frac{1}{2}\left[\sum_{\substack{m=0\\ n\neq 0}} \frac{y^s}{\|n\|^{2s}} + \sum_{\substack{m\neq 0\\ n\in\mathbb{Z}}} \frac{y^s}{|mz+n|^{2s}}\right]$$

$$= y^s \zeta(2s) + \sum_{m>0} \sum_{n\in\mathbb{Z}} \frac{y^s}{|mz+n|^{2s}}$$

$$= y^s \zeta(2s) + y^s \sum_{m>0} \sum_{n\in\mathbb{Z}} f(n)$$

with $f: \mathbb{R} \to \mathbb{R}$, $f(v) := \frac{1}{|mz+v|^{2s}}$

Recall Poisson summation: $f : \mathbb{R} \to \mathbb{R}$ "nice"

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n) \quad \text{with}$$

$$\hat{f}(n) = \int_{-\infty}^{\infty} f(v) e^{-2\pi i n v}\, dv.$$

Hence $\displaystyle\sum_{m>0}\sum_{n\in\mathbb{Z}} f(n) = \sum_{m>0}\sum_{n\in\mathbb{Z}} \hat{f}(n)$

$$= \sum_{m>0}\sum_{n\in\mathbb{Z}} \int_{-\infty}^{\infty} \frac{e^{-2\pi i n v}}{\|mz+v\|^{2s}}\, dv.$$

We now look at this last integral

$$\int_{-\infty}^{\infty} \frac{e(-nv)}{\|mz+v\|^{2s}}\, dv = \int_{-\infty}^{\infty} \frac{e(-nv)}{((mx+v)^2+m^2y^2)^s}\, dv.$$

let $u = \left(\frac{mx+v}{my}\right)$ $\qquad my\,u = mx+v$

$$= \int_{-\infty}^{\infty} \frac{e^{-2\pi i n(myu-mx)}}{(u^2+1)^s (my)^{2s}} \cdot my\, du \qquad (m>0, y>0$$

$$= e^{2\pi i n m x} (my)^{1-2s} \boxed{\int_{-\infty}^{\infty} \frac{e^{-2\pi i n m y u}}{(u^2+1)^s}\, du}$$

$$:= I(-2\pi n m y)$$

where $I(a) := \int_{-\infty}^{\infty} \frac{e^{iau}}{(u^2+1)^s} du$

We have the following lemma for the evaluation of this integral.

<u>Lemma 7.12</u> If $\operatorname{Re} s > \frac{1}{2}$, $a \in \mathbb{R}$ then

$$\Gamma(s) I(a) = \begin{cases} \sqrt{\pi}\, \Gamma(s-\frac{1}{2}) & \text{if } a = 0 \\ 2\sqrt{\pi} \left|\frac{a}{2}\right|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(|a|) & \text{if } a \neq 0 \end{cases}$$

where $K_s(a) := \frac{1}{2} \int_0^{\infty} e^{-a/2\,(t+t^{-1})/2}\, t^{s-1}\, dt$ $\quad a > 0.$

is the K Bessel function

<u>Pf</u>: (Exercise) Idea:
Note if $y > 0$ the integrand in the defn of $K_s(y)$ decays rapidly as $t \to 0$ and $t \to \infty$ and the integral is conv. for all s.

Use $\Gamma(s) = \int_0^{\infty} e^{-t} t^s \frac{dt}{t}$ to write

$$\Gamma(s) I(a) = \int_{-\infty}^{\infty} \frac{e^{+iau}}{(u^2+1)^s} du \int_0^{\infty} e^{-t} t^s \frac{dt}{t} \qquad \operatorname{Re} s > 0.$$

$$= \int_{-\infty}^{\infty} e^{+iau} \left((u^2+1)^{-s} \int_0^{\infty} e^{-t} t^s \frac{dt}{t} \right) du.$$

$$= \int_{-\infty}^{\infty} e^{+iau} \left(\int_0^{\infty} e^{-(u^2+1)t} t^s \frac{dt}{t} \right) du$$

$$\left(\int_0^{\infty} e^{-At} t^s \frac{dt}{t} = A^{-s}\,\Gamma(s) \right)$$

Now interchange the order of $\int$.

$$= \int_0^{\infty} \int_{-\infty}^{\infty} e^{iau} e^{-(u^2+1)t} t^s \, du \, \frac{dt}{t}$$

$$= \int_0^{\infty} e^{-t} t^s \left(\int_{-\infty}^{\infty} e^{iau} e^{-u^2 t} du \right) \frac{dt}{t}$$

Now do the u integral first (which is a F. transform integral of the an exponential).

$\therefore$

Using lemma 7.12 then gives the expansion

$$G(z,s) = y^s \zeta(2s) + \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} \sqrt{\pi}\, y^{1-2s} \zeta(2s)$$

$$+ \frac{2\pi^s}{\Gamma(s)} \sum_{m>0} \sum_{n\neq 0} m^{1-2s} \sqrt{y}\, |mn|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi |nm| y)\, e(n$$ [illegible]

$$= a_0(y,s) + \frac{2\pi^s}{\Gamma(s)} \sum_{n\neq 0} \sum_{d|n} \sqrt{y}\, d^{1-2s} |n|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|n|y)\, e(nx)$$

$$= a_0(y,s) + \frac{2\pi^s}{\Gamma(s)} \sum_{n\neq 0} \sigma_{1-2s}(|n|)\, |n|^{s-\frac{1}{2}} \sqrt{y}\, K_{s-\frac{1}{2}}(2\pi|n|y)\, e(nx)$$

As a cor. we have

Thm 7-13 Let $E(z,s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} (\operatorname{Im} \gamma z)^s$, $\operatorname{Re} s > 1$

and $E^*(z,s) = \frac{1}{2}\pi^{-s}\Gamma(s)\,\zeta(2s)\,E(z,s)$

$= \pi^{-s}\Gamma(s)\, G(z,s)$.

Then $E^*(z,s)$ has a meromorphic continuation to the whole s-plane with simple poles at $s=0,1$ with residues indep of z. Moreover

$$E^*(z,s) = E^*(z,1-s)$$

and $E^*(z,s) = O(y^\sigma)$ as $y \to \infty$ where

$$\sigma = \max\{\operatorname{Re} s, \operatorname{Re} 1-s\}.$$

Proof The F.exp in Thm 7-11 of $G(z,s)$ gives

$$E^*(z,s) = \pi^{-s}\Gamma(s)\zeta(2s)y^s + \pi^{-s+\frac{1}{2}}\Gamma(s-\tfrac{1}{2})\zeta(2s-1)y^{1-s}$$

$$+ 2\sum_{n\neq 0} |n|^{s-\frac{1}{2}}\sigma_{1-2s}(|n|)\sqrt{y}\,K_{s-\frac{1}{2}}(2\pi|n|y)\,e(nx).$$

$$E^*(z,s) = \sum a_n^*(y,s)\,e(nx) \quad w/$$

$$a_0^*(y,s) = \pi^{-s}\Gamma(s)\zeta(2s)y^s + \pi^{\frac{1}{2}-s}\Gamma(s-\tfrac{1}{2})\zeta(2s-1)$$

$$n\neq 0 \quad a_n^*(y,s) = 2|n|^{s-\frac{1}{2}}\sigma_{1-2s}|n|\sqrt{y}\,K_{s-\frac{1}{2}}(2\pi|n|y)$$

$$a_0^*(y,s) = \Lambda(2s)y^s + \Lambda(2-2s)y^{1-s}$$

$$\text{with} \quad \Lambda(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(2s) = \Lambda(1-s)$$

$$\text{Hence} \quad a_0^*(y,1-s) = a_0^*(y,s)$$

$a_0^*(y,s)$ has simple poles at $s=0$, $s=\frac{1}{2}$ and $s=1$. But the residue at $s=\frac{1}{2}$ is equal to zero since residues of 2 terms cancel each other.

Each $a_n^*(y,s)$ has A.C to all s since $K_s(y)$ is analy for all s.

The convergence of the infinite series $\sum_{n\neq 0} a_n^*(y,s)e(nx)$ follows from the rapid decay of $K_s(y)$

which is given in the following lemma which collects several properties of $K_s(y)$

<u>Lemma</u> $K_s(y) = \frac{1}{2}\int_0^\infty e^{-y(t+1/t)/2} t^s \frac{dt}{t}$

satisfies

(a) The integral on the RHS converges $\forall s$

(b) $|K_s(y)| \le e^{-y/2} K_{\operatorname{Re} s}(2)$ if $y > 4$

(c) $K_s(y) = K_{-s}(y)$

(d) $K_s(y)$ satisfies the O.D.E.

$$y^2 u'' + y u' - (y^2 + s^2) u = 0.$$

<u>Proof</u> (Idea) (a) This we've seen already, due to the decay of the integrand as $y \to 0$ or $y \to \infty$.

(b) If $a, b > 2$ then $ab > a + b$ and $e^{-ab} < e^{-(a+b)}$

Apply this w/ $a = y/2$ $b = t + t^{-1}$

$$|K_s(y)| \le e^{-y/2} \frac{1}{2} \left|\int_0^\infty e^{-(t+t^{-1})} t^s \frac{dt}{t}\right| = e^{-y/2} K_{\operatorname{Re} s}(2)$$

(c) The integrand is inv. under $t \to t^{-1}$ as $s \to -s$

(d) exercise

Using the last lemma, we conclude the convergence of the F. expansion which gives then the A-C of $E^*(z,s)$ with poles at $s=0, 1$.

To see the F.E, we note $a_n(y,s) = a_n(y, 1-$ $\forall n$.

for $n=0$, This follows from $\Lambda(s)=\Lambda($

For $n \neq 0$ $\quad K_{-w}(y) = K_w(y)$ and

$$n^w \sigma_{-2w}(n) = \sum_{d_1 d_2 = n} d_1^{-w} d_2^{-w} = n^{-w} \sigma_{2w}(n)$$

Hence $a_n(y,s) = 2\sqrt{y}\, |n|^{s-\frac{1}{2}} \sigma_{1-2s}(|n|) K_{s-\frac{1}{2}}(2\pi|n|$

$$= 2\sqrt{y}\, |n|^{1/2-s} \sigma_{2s-1} |n| K_{\frac{1}{2}-s}(2\pi|n|y$$

$$= a_n(y, 1-s)$$

Hence

$$E^*(z,s) = E^*(z, 1-s)$$

Finally the growth of $E^*(z,s)$ follows since the non-constant coef terms decay rapidly as $y \to \infty$. The asymptotic behavior is same as $a_0(y,s) = * y^s + * y^{1-s}$

hence $E^*(z,s) = O(y^\sigma)$ $\quad \sigma = \max\{\mathrm{Re}\, s, \mathrm{Re}(1-s$

Cor $G(z,s)$ has merom-continuation to all $s \in \mathbb{C}$ with a simple pole at $s=1$ and residue $\pi/2$. In particular the residue of $G(z,s)$ at $s=1$ is indep of z.

Pf This follows from calculation of residue of $a_0^*(y,s)$ at $s=1$

As before we have class # formula as a

Cor For $D<-4$

$$\zeta_D(s) = \left(\frac{|D|}{4}\right)^{-s/2} \sum_{Q \in \mathcal{Q}_D / P} G(z_Q, s)$$

has A.C to all s-plane w/ simple pole at $s=1$ and residue

$$\operatorname*{Res}_{s=1} \zeta_D(s) = \left(\frac{|D|}{4}\right)^{-1/2} \frac{\pi}{2} h_D = \frac{\pi h_D}{\sqrt{|D|}}$$

$\square$

The Eisenstein series is an example of Maass form of wt 0 for $\Gamma = SL(2, \mathbb{Z})$

<u>Defn</u> Let $\Gamma \leq SL(2, \mathbb{Z})$. A function

$f : \mathbb{H} \to \mathbb{C}$ is called <u>Γ automorphic</u>

if $f(\gamma z) = f(z) \quad \forall \gamma \in \Gamma, \quad z \in \mathbb{H}$

f is called a Maass form for Γ

if the following is satisfied.

(i) $f(\gamma z) = f(z) \quad \forall \gamma \in \Gamma$

(ii) $(\Delta + \lambda) f = 0$ with $\lambda \in \mathbb{C}$

$$\Delta = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

(iii) $f(\sigma_{\mathfrak{a}}(x+iy)) \ll y^A$ for any cusp $\mathfrak{a}$,

$\forall x \in \mathbb{R}, \; y \geq 1$ for some constant $A \geq 0$.

where $\sigma_{\mathfrak{a}} \in SL_2(\mathbb{R})$ s.t $\sigma_{\mathfrak{a}}(\infty) = \mathfrak{a}$

$$\sigma_{\mathfrak{a}}^{-1} \Gamma_{\mathfrak{a}} \sigma_{\mathfrak{a}} = \Gamma_\infty \qquad \mathfrak{a} \in \mathbb{Q} \cup \{\infty\} / \Gamma.$$

a cusp of Γ.

Rmk. Condition (iii) is a moderate growth condition

For simplicity we restrict to the case $\Gamma = SL_2(\mathbb{Z})$ so that we have only one cusp.

$\mathcal{A}(\Gamma)$ = space of Autom. functions

$\mathcal{M}(\Gamma)$ = space of Maass forms.

For any $f \in \mathcal{M}(\Gamma)$, since $f(z+1) = f(z)$

$$f(x+iy) = \sum_{n \in \mathbb{Z}} a_n(y) e(nx)$$

If f has eigenvalues $\lambda = s(1-s)$, $s \in \mathbb{C}$

then assuming we can differentiate the Fourier series termwise the eigenfunction condition

$$\Delta f = s(s-1) f \quad \text{becomes}$$

$$\sum_n (2\pi i n)^2 y^2 a_n(y) e(nx) + y^2 a_n''(y) e(nx)$$
$$= s(s-1) \sum a_n(y) e(nx)$$

ie for each n

$$-4\pi n^2 y^2 a_n(y) + y^2 a_n''(y) = s(s-1) a_n(y)$$

or equivalently

$$a_n'' + \left(\frac{s(1-s)}{y^2} - 4\pi^2 n^2\right) a_n = 0$$

which as 2 lin. indep. solns

$$K(y) = \begin{cases} y^{1/2} \\ y^{1/2}\log y \end{cases} \quad \text{if } n=0, \ s=\tfrac{1}{2}$$

$$= \begin{cases} y^{s} \\ y^{1-s} \end{cases} \quad \text{if } n=0 \quad s \neq \tfrac{1}{2}$$

$$= \begin{cases} y^{1/2} \; K_{s-\frac{1}{2}}(2\pi|n|y) \\ y^{1/2} \; I_{s-\frac{1}{2}}(2\pi|n|y) \end{cases} \quad \text{if } n \neq 0$$

This last soln is excluded if we require $f(z)$ to grow no more than $O(y^A)$ at ∞

We then have

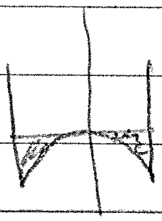
Thm Any $f \in \mathcal{M}(\Gamma)$ Maass form of e. value $\lambda = s(1-s)$ has F. exp.

$$f(z) = a_0^+ y^s + a_0^- y^{1-s} + \sum_{n \neq 0} a_n W_s(nz)$$

where $W_s(z) = 2\sqrt{y}\, K_{s-\frac{1}{2}}(2\pi |n| y)\, e(nx)$.

The growth condition (iii) in the defn of $\mathcal{M}(\Gamma)$; $f(z) \ll y^A$ for $y > 1$

By compactness of $z \in \mathcal{F} \setminus \{ z \mid |x| < \frac{1}{2}, y > 1 \}$



and continuity of $f(z)$, $f(z) \ll 1$ in $z \in \mathcal{F} \setminus \{ \dots \}$

Hence for any $z \in \mathbb{H}$

$$f(z) \ll 1 + y^A \ll y^A + y^{-A}$$

As in the holom. case this gives

<u>Lemma</u> If $f \in \mathcal{M}_s(\Gamma)$ then $a_n \ll |n|^A$

<u>Pf</u> $f(z) = a_0^+ y^s + a_0^- y^{1-s} + \sum_{n \neq 0} a_n W_s(nz)$

$$a_n W_s(iny) = \int_0^1 f(z)\, e(-nx)\, dx \ll y^A + y^{-A}$$

for any $y > 0$. Taking $y = n^{-1}$ gives the result ☐

Inserting this bound into the F.exp gives $f(z) = a_0^+ y^s + a_0^- y^{1-s} + O(e(-2\pi y))$ for $y > 1$. In particular condition (iii) holds true with $A = \max\{\operatorname{Re} s, \operatorname{Re}(1-s)\}$ where $\lambda = s(1-s)$ is the e.value of f

<u>Defn</u> A Maass form $f(z)$ is called a <u>cusp form</u> for Γ if $a_0^+ = a_0^- = 0$

In that case

<u>Lemma</u> If f is a Maass cusp form then

$f(z) \ll e^{-2\pi y}$ $\quad z \in \mathbb{H}$ thus f

is bounded in $\mathbb{H}$.

Let $C(\Gamma \backslash \mathbb{H}) = \{ f \in \mathcal{A}(\Gamma) \mid f$ smooth, bdd, $a_0(f, y) = 0 \}$

$C_s(\Gamma \backslash \mathbb{H}) = \{ f \in C(\Gamma \backslash \mathbb{H}) \mid \Delta f = s(\cancel{1-s}) f \}$

cuspidal Maass forms.

Then

<u>Thm</u> $C(\Gamma \backslash \mathbb{H})$ is spanned by cusp forms

Let $u_0 = \sqrt{\frac{3}{\pi}}$ constant function

$\{u_j\}_{j \geq 1}$ an orthonormal basis of $C(\Gamma \backslash \mathbb{H})$

ordered by increasing eigenvalue λ_j

ie $\Delta u_j = \lambda_j u_j$ and

$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \cdots \leq \lambda_n \cdots$

Then $u \in C(\Gamma \backslash \mathbb{H})$ has the expansion

$u(z) = \sum_{j \geq 0} \langle u, u_j \rangle u_j(z)$. where $\langle u, u_j \rangle = \int_{\Gamma \backslash \mathbb{H}} u(z) \overline{u_j(z)} \frac{dx\,dy}{y^2}$

Note The Eisenstein series is not a cusp form but is an eigenfunction of Δ

This follows from

(1) $G(z,s) = \zeta(2s)\, E(z,s)$

where $E(z,s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} (\operatorname{Im} \gamma z)^s = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} y^s | \gamma$

(2) Δ commutes with the action of Γ on functions $f : \mathbb{H} \to \mathbb{C}$

ie $(\Delta f) | \gamma = \Delta (f | \gamma)$

where $((\Delta f) | \gamma)(z) = (\Delta f)(\gamma z)$

$(\Delta (f | \gamma))(z) = \Delta (f(\gamma z))$

Hence $\Delta E(z,s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \Delta (y^s | \gamma)$

$= \sum (\Delta y^s) | \gamma$

Since $\Delta y^s = s(s-1) y^s$

$= s(s-1) \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} y^s | \gamma$